Conservative Vector Fields

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1 Vector Fields

We consider a **region** $\mathcal{R}$ in two-dimensional space $\mathbb{R}^2$ or in three-dimensional space $\mathbb{R}^3$.

The region $\mathcal{R}$ is **open** if it does not contain any of its boundary points.

A **curve** $\mathcal{C}$ in the region $\mathcal{R}$ is a set of continuous points in $\mathcal{R}$ from an initial point to an end point. It has no gaps or discontinuities.

A curve $\mathcal{C}$ is said to be **closed** if its initial and end points are the same point. For example, a circle is a closed curve.

A curve $\mathcal{C}$ is **simple** if it does not cross or touch itself. An ellipse is a simple curve. The curve that displays the number 8 is not simple.

In $\mathbb{R}^3$, a curve is given by equations of the form $x = x(t),\ y = y(t),\ z = z(t)$ for the coordinates (x, y, z) of any point on the curve. Here, t is parameter and different values of t give different points on the curve. In $\mathbb{R}^2$, we omit the third equation.

We say that a curve is **smooth** if the derivatives $\dfrac{dx}{dt}, \dfrac{dy}{dt}$ and $\dfrac{dz}{dt}$ exist and are continuous at all points on the curve and, also $\left(\dfrac{dx}{dt}\right)^2 + \left(\dfrac{dy}{dt}\right)^2 + \left(\dfrac{dz}{dt}\right)^2 > 0$ at all interior points. These conditions ensure that the curve has a tangent line at each point on it and the slope is continuous.

A continuous curve is said to be **piecewise smooth** if it is smooth everywhere except at a finite number of points. Although, here, we focus on results for smooth curves, the results that we consider are generally valid for piecewise curves.

We use the term *path* to mean a simple piecewise smooth curve.

We say that a region $\mathcal{R}$ is **connected** if *any* two points in the region can be connected by a curve that lies entirely in $\mathcal{R}$.

A region $\mathcal{R}$ is **simply-connected** if it is *connected* and *every* closed curve in the region can be shrunk through that region to a point in the region. Such a region cannot contain any holes or gaps. For example, a sphere is simply connected. A torus is not simply connected

We say that $\boldsymbol{F}$ is a **vector field** over a region $\mathcal{R}$ in two-dimensional space $\mathbb{R}^2$ or in three-dimensional space $\mathbb{R}^3$ if it is a vector function that assigns to each point $P(x, y)$ in $\mathbb{R}^2$ or $P(x, y, z)$ in $\mathbb{R}^3$ a vector given by $\boldsymbol{F}(x, y)$ or $\boldsymbol{F}(x, y, z)$.

Examples of applications of vector fields in physics are electric fields, magnetic fields, gravitational fields and velocity fields in fluid flows.

2 Some Useful Results

Here, we note some useful results that are required to establish the equivalence of the different definitions that may be given for conservative vector fields.

First, we have the identity

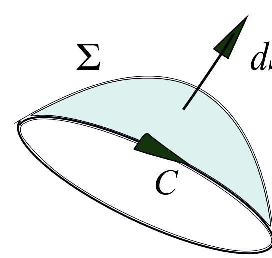
$$\operatorname{curl}(\nabla\phi) \equiv \mathbf{0} \tag{1}$$

for a continuously differentiable scalar function ϕ over any region $\mathcal{R}$.
That is, the curl of the gradient of a continuously differentiable scalar function over any region is the zero vector at all points of that region.

We require Stokes' theorem, which reads

$$\oint_{\mathcal{C}} \boldsymbol{F} \cdot d\boldsymbol{r} = \iint_{\Sigma} \operatorname{curl} \boldsymbol{F} \cdot d\boldsymbol{S}\,, \tag{2}$$

where, $\mathcal{C}$ is any simple closed curve in a simply connected region $\mathcal{R}$.
Σ is any open smooth surface in $\mathcal{R}$ with $\mathcal{C}$ as its boundary.
They are oriented so that the positive direction of $\mathcal{C}$ and the positive direction of $d\boldsymbol{S}$ form a right-handed system.
The theorem says that the circulation of $\boldsymbol{F}$ around any closed curve $\mathcal{C}$ is equal to the flux of $\operatorname{curl}\boldsymbol{F}$ across any surface Σ that spans $\mathcal{C}$ and is appropriately oriented.



Next, let $\mathcal{C}$ be a smooth curve from a point A to a point B that is given parametrically by $\boldsymbol{r} = \boldsymbol{r}(t)$ for $a \le t \le b$ in a simply connected region $\mathcal{R}$. Here, a and b are the values of t at the points A and B, respectively. We note that $x = x(t),\ y = y(t),\ z = z(t)$ along $\mathcal{C}$. Let $f = f(\boldsymbol{r})$ be a continuously differentiable function of position defined on $\mathcal{R}$. Then, we can write

$$\int_{\mathcal{C}} \nabla f \cdot d\boldsymbol{r} = \int_a^b \left(\nabla f \cdot \frac{d\boldsymbol{r}}{dt}\right) dt = \int_a^b \left(\frac{df}{dt}\right) dt = \int_{f(a)}^{f(b)} df = f_B - f_A\,, \tag{3}$$

where $f_A = f(a)$ and $f_B = f(b)$ are the values of f at the points A and B, respectively. That is, the integral depends only on the values of f at the end points and is independent of the path.

3 Definitions

We consider four alternate definitions for a vector field $\boldsymbol{F}$ with continuous partial derivatives over some region $\mathcal{R}$ to be a **conservative** vector field. If we choose any one of them as our basic definition of a conservative vector field, then the remaining three can be established as results that can be derived from the basic definition.

The alternative definitions are as follows:

Definition 1. *$\boldsymbol{F}$ is a conservative vector field over the region $\mathcal{R}$ if there is a continuously differentiable scalar function ϕ defined over $\mathcal{R}$ such that $\boldsymbol{F} = \nabla\phi$.*

The function ϕ is called a ***potential function*** *for the vector field $\boldsymbol{F}$.*

Definition 2. *$\boldsymbol{F}$ is a conservative vector field over a simply connected region $\mathcal{R}$ if curl $\boldsymbol{F} = \boldsymbol{O}$ at every point of $\mathcal{R}$.*

A vector satisfying this condition is said to be ***irrotational*** *on $\mathcal{R}$.*

Definition 3. *$\boldsymbol{F}$ is a conservative vector field over a simply connected region $\mathcal{R}$ if the line integral $\oint_{\mathcal{C}} \boldsymbol{F} \cdot d\boldsymbol{r} = 0$ for* every simple closed curve *$\mathcal{C}$ in $\mathcal{R}$.*

The line integral of a vector field $\boldsymbol{F}$ around a closed curve $\mathcal{C}$ is called the ***circulation*** *of $\boldsymbol{F}$ around $\mathcal{C}$.*

Definition 4. *$\boldsymbol{F}$ is a conservative vector field over a simply connected region $\mathcal{R}$ if the line integral $\int_{\mathcal{C}} \boldsymbol{F} \cdot d\boldsymbol{r}$ is* independent of the path *$\mathcal{C}$ from A to B for* every pair of points *$\{A, B\}$ in $\mathcal{R}$.*

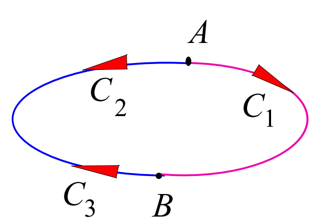
4 Equivalence of the Definitions

In this section, we indicate how the remaining three definitions may be derived as results when one of the four definitions is adopted as the basic one.

Definition 1 $\rightarrow$ 2:
If $\boldsymbol{F} = \nabla\phi$, on the region $\mathcal{R}$ then the identity (1) implies that curl $\boldsymbol{F} = \boldsymbol{O}$ on $\mathcal{R}$.

Definition 2 $\rightarrow$ 3:
If curl $\boldsymbol{F} = \boldsymbol{O}$ on the region $\mathcal{R}$, then Equation (2) in Stokes' theorem implies that $\oint_{\mathcal{C}} \boldsymbol{F} \cdot d\boldsymbol{r} = 0$ for *every simple closed curve* $\mathcal{C}$ in $\mathcal{R}$.

Definition 3 $\rightarrow$ 4:
Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two piecewise smooth curves from a point A to point B in a simply connected region $\mathcal{R}$.
Let $\mathcal{C}_3$ denote the curve from B to A taken along $\mathcal{C}_2$ but in the opposite direction.
Then, $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_3$ is a closed curve that is piecewise smooth.
Since $\oint_{\mathcal{C}} \boldsymbol{F} \cdot d\boldsymbol{r} = 0$ we have $\int\limits_{\mathcal{C}_1} \boldsymbol{F} \cdot d\boldsymbol{r} + \int \mathcal{C}_3 \boldsymbol{F} \cdot d\boldsymbol{r} = 0,.$



This gives $\int\limits_{\mathcal{C}_1} \boldsymbol{F} \cdot d\boldsymbol{r} = -\int\limits_{\mathcal{C}_3} \boldsymbol{F} \cdot d\boldsymbol{r} = \int\limits_{\mathcal{C}_2} \boldsymbol{F} \cdot d\boldsymbol{r}$
Thus, $\int \boldsymbol{F} \cdot d\boldsymbol{r}$ taken along $\mathcal{C}_1$ from A to B is equal to $\int \boldsymbol{F} \cdot d\boldsymbol{r}$ along $\mathcal{C}_2$ from A to B.
This implies that $\int \boldsymbol{F} \cdot d\boldsymbol{r}$ is independent of the path from A to B because we can choose $\mathcal{C}_1$ and $\mathcal{C}_2$ to be any two curves from A to B.
Also, this is true for any pair of points A and B in $\mathcal{R}$.

Definition 4 $\rightarrow$ 1:
We know that $\int_{\mathcal{C}} \boldsymbol{F} \cdot d\boldsymbol{r}$ from any point A in the region $\mathcal{R}$ to any other point B in $\mathcal{R}$ is independent of the path $\mathcal{C}$ from A to B. We select a fixed point $P_\circ(\boldsymbol{r}_\circ)$ in $\mathcal{R}$. Let $P(\boldsymbol{r})$ be any general point in $\mathcal{R}$ and let $\mathcal{C}$ be any smooth curve from $P_\circ$ to P with points defined by the parameter t so that $\boldsymbol{r} = \boldsymbol{r}(t)$ for points along $\mathcal{C}$.

We define the function $\phi(\boldsymbol{r})$ of position $\boldsymbol{r}$ as

$$\phi(\boldsymbol{r}) = \int_{\mathcal{C}} \boldsymbol{F} \cdot d\boldsymbol{r} = \int_{t_\circ}^{t} \boldsymbol{F} \cdot \frac{d\boldsymbol{r}}{dt} dt\,. \tag{4}$$

The Fundamental theorem of Calculus gives $\dfrac{d\phi}{dt} = \boldsymbol{F} \cdot \dfrac{d\boldsymbol{r}}{dt}$.

Also, from the chain rule for partial derivatives we know that $\dfrac{d\phi}{dt} = \nabla\phi \cdot \dfrac{d\boldsymbol{r}}{dt}$.

These two equations require that $(\boldsymbol{F} - \nabla\phi) \cdot \dfrac{d\boldsymbol{r}}{dt} = 0$ at all points along $\mathcal{C}$.

Since this result is true for any smooth curve in $\mathcal{R}$, we must have $\boldsymbol{F} = \nabla\phi$ at all points in $\mathcal{R}$.

Because we have established the following cycle of results $1 \to 2 \to 3 \to 4$, we can claim that any one of four the definitions is equivalent to each of the remaining definitions.

In some cases, it will be easier to derive the result in a definition directly from another definition without using the cycle of results. For example, Equation (3) directly establishes that the result in Definition 4 follows from Definition 1.

Note 4.1. *We can develop other equivalent definitions from the four definitions that are given above.*

For example, we can combine Definitions 3 and 4 to obtain the following definition.

Definition 5. $\boldsymbol{F}$ *is a conservative vector field over a simply connected region* $\mathcal{R}$ *if, for* every pair of points $\{A, B\}$ *in* $\mathcal{R}$, $\int_{\mathcal{C}_1} \boldsymbol{F} \cdot d\boldsymbol{r} = -\int_{\mathcal{C}_2} \boldsymbol{F} \cdot d\boldsymbol{r}$, *where* $\mathcal{C}_1$ *is any path from* A *to* B *and* $\mathcal{C}_2$ *is any path from* B *to* A *.*

5 Applications

There are many physical phenomena that are described using conservative vector fields. Some examples are (i) non-relativistic gravitational field $\boldsymbol{g}$, (ii) electrostatic field $\boldsymbol{E}$, (iii) magnetostatic field $\boldsymbol{B}$, and (iv) velocity field $\boldsymbol{v}$ of an irrotational flow of a fluid. The first two are examples of force fields in that $\boldsymbol{g}$ times the mass m or $\boldsymbol{E}$ times the electric charge q of a small object represents a force on the object.

5.1 Force Fields

Let $\boldsymbol{F} = \nabla\phi$ be a conservative force field with potential function ϕ over some simply connected region $\mathcal{R}$. Suppose that the force $\boldsymbol{F}$ acts on some small object represented by a point. Then, the work done by the force field on the object when it moves from point A to point B is given by

$$W = \int_A^B \boldsymbol{F} \cdot d\boldsymbol{r}\,. \tag{5}$$

This definition together with Equation (3) gives

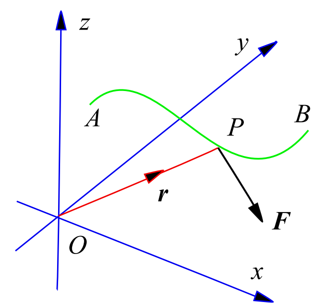
$$W = \int_A^B \boldsymbol{F} \cdot d\boldsymbol{r} = \int_A^B \nabla f \cdot d\boldsymbol{r} = \phi_B - \phi_A\,. \tag{6}$$

That is, the work done by a conservative force on an object is equal to the change in potential.

Definition 3 implies that when an object moves around a closed curve and returns to its starting point under the action of a conservative force field in a simply connected region, the total work done by the force field is zero.

Definition 4 implies that when an object moves from a point A and to a point B under the action of a conservative force field in a simply connected region, the total work done by the force field is independent of the path taken from A to B.

5.2 Conservation of Energy of a Moving Particle



Suppose that a particle of fixed mass m moves under the action of a conservative force field $\boldsymbol{F}$ in a simply connected region $\mathcal{R}$ along a curve $\mathcal{C}$ from the point A to the point B.
Let the time $t = a$ when the object is at A and $t = b$ at B.
We use an inertial frame of reference with origin at the point O.
At time t, let the particle be at the point P with position vector $\boldsymbol{r}$.
That is, $\boldsymbol{r} = \boldsymbol{r}(t)$.

The velocity $\boldsymbol{v}$ and acceleration $\boldsymbol{a}$ of the particle are given by $\boldsymbol{v} = \dfrac{d\boldsymbol{r}}{dt}$ and $\boldsymbol{a} = \dfrac{d\boldsymbol{v}}{dt} = \dfrac{d^2\boldsymbol{r}}{dt^2}$.

Let $|\boldsymbol{v}| = v$. Then, $\boldsymbol{v}\cdot\boldsymbol{v} = v^2$ and $\dfrac{d}{dt}(\boldsymbol{v}\cdot\boldsymbol{v}) = \dfrac{d}{dt}(v^2)$. Thus, $\boldsymbol{v}\cdot\dfrac{d\boldsymbol{v}}{dt} = \dfrac{d}{dt}\left(\dfrac{v^2}{2}\right)$.

Newton's law of motion requires that $\boldsymbol{F} = m\boldsymbol{a} = m\dfrac{d\boldsymbol{v}}{dt}$.
The work done by the force field on the object when it moves from A to B along the path $\mathcal{C}$ is

$$W = \int_{\mathcal{C}} m\frac{d\boldsymbol{v}}{dt}\cdot d\boldsymbol{r} = \int_a^b m\frac{d\boldsymbol{v}}{dt}\cdot\frac{d\boldsymbol{r}}{dt}dt = \int_a^b m\,\boldsymbol{v}\cdot\frac{d\boldsymbol{v}}{dt}dt = \int_a^b \frac{d}{dt}\left(\frac{m\,v^2}{2}\right)dt = \left(\frac{m\,v^2}{2}\right)\Bigg|_{t=a}^{t=b}, \tag{7}$$

which gives

$$W = \frac{m\,v_B^2}{2} - \frac{m\,v_A^2}{2}\,, \tag{8}$$

where v_A and v_B are the speeds of the object at the points A and B, respectively.
The quantity $\dfrac{mv^2}{2}$ is the **kinetic energy** of the object.
Equation (8) gives us the following theorem:
The work done by a conservative force on an object is equal to the change in kinetic energy of the object.

In the study of mechanics we define the concept of the **potential energy** V of an object in a force field by the equation $\boldsymbol{F} = -\nabla V$.
That is, $V = -\phi$, where ϕ is the potential function that we used in the definition of a conservative vector field. From Equation (4) we have

$$V(\boldsymbol{r}) = -\int_{\mathcal{C}} \boldsymbol{F}\cdot d\boldsymbol{r}\,, \tag{9}$$

where C is any smooth curve from a fixed point $P_\circ(\boldsymbol{r}_\circ)$ to the point $P(\boldsymbol{r})$.

We can combine Equations (6) and (8) to obtain the result

$$\frac{m\,v_B^2}{2} + V_B = \frac{m\,v_A^2}{2} + V_A\,. \tag{10}$$

That is, the sum of the kinetic energy and the potential energy is constant.
This is the **Law of Conservation of Energy** for the dynamics of an object in a conservative force field.

6 Examples

6.1 A General Example

Let $\boldsymbol{F} = (2xy + yz + e^{5x})\boldsymbol{i} + (x^2 + 3y^2 + xz)\boldsymbol{j} + [xy + 4\cos(2z)]\boldsymbol{k}$ be a force field,
where $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ are base vectors in a rectangular coordinate system. We use Gaussian units so that force is given in dynes and length in centimeters.

(a). We use Definition 2 to show that $\boldsymbol{F}$ is conservative over $\mathbb{R}^3$.
We have $\text{curl}\boldsymbol{F} = (x - x)\boldsymbol{i} - (y - y)\boldsymbol{j} + [(2x + z) - (2x + z)]\boldsymbol{k} = \boldsymbol{O}$.
This is true for all values of (x, y, z). So, $\boldsymbol{F}$ is conservative on $\mathbb{R}^3$.

(b)We find the potential function ϕ such that $\boldsymbol{F} = \nabla\phi$ as follows, where ϕ is given in ergs.
ϕ must satisfy $\dfrac{\partial\phi}{\partial x} = 2xy + yz + e^{5x}$, $\dfrac{\partial\phi}{\partial y} = x^2 + 3y^2 + xz$, $\dfrac{\partial\phi}{\partial z} = xy + 4\cos(2z)$.

Integrating the first of these with respect to x gives $\phi(x, y, z) = x^2y + xyz + \dfrac{e^{5x}}{5} + \psi(y, z)$,
where $\psi(y, z)$ is a function of y and z.

This requires $\dfrac{\partial\phi}{\partial y} = x^2 + xz + \dfrac{\partial\psi}{\partial y} = x^2 + 3y^2 + xz$.

That is, $\dfrac{\partial\psi}{\partial y} = 3y^2$ and $\psi = y^3 + \xi(z)$, where ξ is a function of z.

Now, we have $\phi(x, y, z) = x^2y + xyz + \dfrac{e^{5x}}{5} + y^3 + \xi(z)$.

This gives, $\dfrac{\partial\phi}{\partial z} = xy + \dfrac{d\xi}{dz} = xy + 4\cos(2z)$ so that $\dfrac{d\xi}{dz} = 4\cos(2z)$.
Then, $\xi = 2\sin(2z) + C$, where C is an arbitrary constant.

Thus, $\phi(x, y, z) = x^2y + xyz + y^3 + \dfrac{e^{5x}}{5} + 2\sin(2z) + C$.

(c). We find the work done, W ergs, by $\boldsymbol{F}$ in moving an object from the point $A(-2, 3, 0)$ to the point $B(5, -4, \pi)$ by using the potential function ϕ and Equation (6).

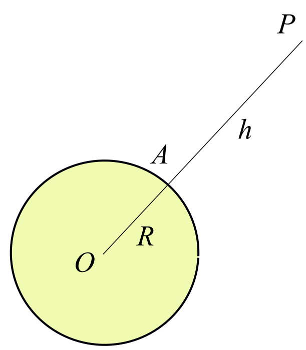
We have $W = (-100 - 20\pi - 64 + \dfrac{e^{25}}{5} + 0 + C) - (12 + 0 + 27 + \dfrac{e^{-4}}{5} + 0 + C)$.

That is $W = \dfrac{e^{25} - e^{-4}}{5} - 203 - 20\pi$.

6.2 Motion under Earth's Gravity

We consider the motion of an object of mass m under earth's gravity. We assume that appropriate units are used.

Let O, M and R denote the center, mass and radius of the earth.
At time t let the object be at the point P and let $\overrightarrow{OP} = \boldsymbol{r}$.
Let the line OP meet the earth's surface at the point A.
Let $h = AP$ be the height of the object at time t. That is $r = R + h$.
The gravitational force per unit mass due to the earth is
$\boldsymbol{g} = -\left(\frac{GM}{r^2}\right)\left(\frac{\boldsymbol{r}}{r}\right)$, where G is the gravitational constant.
The force on the on the object is $\boldsymbol{F} = -\frac{GMm\boldsymbol{r}}{r^3}$.
This is the Inverse Square Law for gravitation postulated by Isaac Newton



$\boldsymbol{F}$ is a conservative force in the region $r > R$ because curl$\boldsymbol{F} = \boldsymbol{O}$ in that region. We can verify this easily by using spherical coordinates.
Let ϕ be the gravitational potential function given by $\boldsymbol{F} = \nabla\phi$.
Since we are using a spherically symmetric gravitational model in which $\boldsymbol{F}$ is a function of r only and acts in a radial direction, ϕ is a function of r only and we have $\frac{d\phi}{dr} = -\frac{GMm}{r^2}$.
Then, $\phi = \frac{GMm}{r} + C$, where C is an arbitrary constant.
We fix the value of C by using the condition that $\phi \to 0$ as $r \to \infty$.
Then, $C = 0$ and we have $\phi = \frac{GMm}{r}$.

Also, from Equation (9), the gravitational potential energy of the object is $V = -\frac{GMm}{r}$.
Let the object be launched with initial speed U from a point on the surface of the earth and let its speed at time t be v. Then, Equation (10) for the conservation of energy gives

$$\frac{m\,v^2}{2} - \frac{GMm}{r} = \frac{m\,U^2}{2} - \frac{GMm}{R}\,. \tag{11}$$

We solve for v to obtain

$$v = \left(U^2 - \frac{2GM}{R} + \frac{2GM}{r}\right)^{1/2}\,. \tag{12}$$

Suppose that we want the object to travel to a large distance from the earth so that we may take the limit as $r \to \infty$. Let the speed in this limit be denoted by v_∞. Then, we have

$$v_\infty = \left(U^2 - \frac{2GM}{R}\right)^{1/2}\,. \tag{13}$$

This equation shows that, if the object is to travel to a large distance, the launch speed U must satisfy the condition $U > \left(\frac{2GM}{R}\right)^{1/2}$.

We put $U_{esc} = \left(\frac{2GM}{R}\right)^{1/2}$.
This is called the **escape velocity** for an object to get away from the earth's gravitational field.