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Analysis of Graphs of Functions

In this section, we apply differential calculus to analyze important properties of the graph of a given function and use this analysis to draw a sketch of the graph. The properties that we consider are (i) extent, (ii) symmetry, (iii) intercepts, (iv) asymptotes; (v) monotonicity, (vi) extrema, (vii) concavity, and (viii) points of inflection.

Suppose that the function $f(x)$ is given. Its graph in the xy-plane is the graph of the equation $y = f(x)$.

A. Extent

The extent of the graph is determined by the domain and range of the function.

Domain

The domain is usually given. If not, we take the assumed domain, which is the set of all possible inputs. That is, the set of all possible x-values.

Range

The range is the set of all possible outputs, or y-values. If the range cannot be determined easily from the form of the function, it can be found when the analysis of the graph is completed and a rough sketch is drawn.

B. Symmetry

We consider symmetry about the y-axis or the origin. We test whether the function is odd, even or neither. If it is neither, the graph does not have any of the usual symmetries.

Symmetry about the y-axis

The graph will be symmetrical about the y-axis if the function is even.
That is, $f(-x) = f(x)$ for all x in the domain.

Symmetry about the origin

The graph will be symmetrical about the origin if the function is odd.
Here, $f(-x) = -f(x)$ for all x in the domain.

C. Intercepts

The y-intercept and the x-intercepts give valuable information about a graph.

y-intercept

We find the y-intercept by setting $x = 0$ in the equation $y = f(x)$ and solving for y.

x-intercepts

The x-intercepts are found by solving the equation $f(x) = 0$ for x.

D. Asymptotes

While polynomial functions do not have linear asymptotes, graphs of rational functions and other composite functions may possess vertical, horizontal or oblique asymptotes. For a more detailed discussion of asymptotes of rational functions you should look up a textbook on college algebra or see the Notes on Graphs of Rational Functions by M. Maheswaran, at
http://mthwww.uwc.edu/wwwmahes/courses/math/mm110nts.htm.

Vertical Asymptotes
The line $x = c$ is a vertical asymptote if $f(x) \to \pm\infty$ as $x \to c$.

Horizontal Asymptotes
The line $y = b$ is a horizontal asymptote if $f(x) \to b$ as $x \to \pm\infty$.

Oblique Asymptotes
The line $y = mx + b$ is an oblique asymptote if $f(x) \to mx + b$ as $x \to \pm\infty$.

E. Monotonicity

A function is said to be monotonic on an interval if it increases throughout the interval or decreases throughout the interval. We use the first derivative $f'(x)$ to determine the monotonicity of $f(x)$. If $f(x)$ increases on an interval then the graph rises on that interval and if $f(x)$ decreases the graph falls.

Critical points for $f'(x)$: To discuss the monotonicity and extrema of functions, we consider the following critical points that involve the first derivative:

1. End points of the domain.
2. Points where $f'(x) = 0$.
3. Points where $f'(x)$ is undefined.

We proceed as follows:

(i) Find the critical points for $f'(x)$.
(ii) Divide the real line into intervals separated by the critical points.
(iii) Set up a sign diagram for $f'(x)$.

$f(x)$ Increases / Graph Rises
On an interval where $f'(x) > 0$, $f(x)$ increases as x increases and the graph rises.

$f(x)$ Decreases / Graph Falls
On an interval where $f'(x) < 0$, $f(x)$ decreases as x increases and the graph falls.

$f(x)$ is Constant
On an interval where $f'(x) = 0$, $f(x)$ is constant and the graph is a horizontal line..

F. Extrema

An extremum is a maximum or minimum. We consider relative (local) extrema and absolute (global) extrema.

Relative or local maxima
These occur at points of the domain where the function changes from increasing to decreasing.

Relative or local minima
These occur at points of the domain where the function changes from decreasing to increasing.

Absolute or global maximum
This is the highest of all the local maxima.

Absolute or global minimum
This is the lowest of all the local minima.

Saddle Points

If $f(x)$ is increasing on both sides of a critical point where $f'(x) = 0$ or decreasing on both sides, then that point is called a saddle point of the graph.

G. Concavity

A graph is concave up on an interval if it lies above its tangent line at every point throughout the interval. In such an interval, $f'(x)$ increases and $f''(x) > 0$.
A graph is concave down on an interval if it lies below its tangent line at every point throughout the interval. In such an interval, $f'(x)$ decreases and $f''(x) < 0$.
We can use the second derivative $f''(x)$ to determine concavity of $f(x)$.

Critical Points for $f''(x)$: To discuss the concavity and inflection points of the graph of a function, we consider the following critical points involving the second derivative:

1. Points where $f''(x) = 0$.
2. Points where $f''(x)$ is undefined.

We proceed as follows: (i) Find the critical points for $f''(x)$. (ii) Divide the real line into intervals separated by the critical points. (iii) Set up a sign diagram for $f''(x)$.

Graph is Concave Up

On an interval where $f''(x) > 0$.

Graph is Concave Down

On an interval where $f''(x) < 0$.

H. Points of Inflection

A point in the domain of a function where the graph changes concavity is called a point of inflection.

Location of Points of Inflection

The location of points of inflection may be determined from the analysis of concavity using the second derivative.

Drawing a Rough Sketch of the Graph

We use the information obtained in the above analysis together with suitably chosen points on the graph to draw a rough sketch. The most convenient points to choose for these selected points are the intercepts and the critical points for both f' and f''. The x-values of these points are known and we can compute the corresponding y-values to set up a table.